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Variational 1/*N* Expansion in Stochastic Quantization with an Auxiliary Field

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Stochastic quantization of the $O(N)$ ϕ^4 scalar field theory leads to a variational determination of the self-energy. An auxiliary composite field is introduced, leading to a simpler formalism. For a solvable toy model, variational results are significantly improved with respect to those without this auxiliary field, even for its crudest propagator approximation.

1. INTRODUCTION

The stochastic quantization method of Parisi and Wu (1981) provides an alternative to conventional canonical and path integral methods to quantize field theories. Euclidean Green functions appear as equilibrium limits of stochastic averages of products of fields with dynamics governed by Langevin equations. Recently our interest focused on the large-N limit in stochastic quantization (Bérard *et al.*, 1995) and on the possibility of variational self-energy expansion in $1/N$ when stochastically quantizing the $O(N)$ ϕ^4 theory (Grandati *et al.*, 1992, 1993). The main purpose of this paper is to present an alternative application of the variational principle. As we shall see, it is possible to introduce an auxiliary stochastic field to eliminate the $O(N)$ ϕ^4 self-interaction. This auxiliary field plays the role of an additional variational parameter and thereby permits a better determination of the self-energy.

The paper is organized as follows: In Section 2 we rapidly recall basics of stochastic quantization and of the variational procedure. Section 3 is devoted to the introduction of an auxiliary stochastic field in this formalism;

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we determine further the different orders in $1/N$ of the variational self-energy of the $O(N)$ ϕ^4 theory in D dimensions. Finally, in Section 4 a completely solvable toy model is used to assess the relevance of the method.

2. STOCHASTIC QUANTIZATION AND VARIATIONAL PRINCIPLE

Let $S_E[\phi]$ be the Euclidean action of some $O(N)$ scalar field in a D-dimensional space-time

$$S_E[\phi] = S_E^0[\phi] + \lambda S_E^{int}[\phi] \tag{1}$$

where $S_E^0[\phi]$ stands for the free action and where the strength λ of the interaction term $S_E^{int}[\phi]$ appears explicitly.

The system is supposed to be in a $(D + 1)$-dimensional heat reservoir, the extra dimension being supplied by a fictitious time t. The equilibrium will be reached in the limit $t \to \infty$. The fields $\phi_\alpha(x, t)$, $\alpha \in \{1, \ldots, N\}$, can now take values in this $(D + 1)$-dimensional Euclidean space and evolve in t according to the Langevin equation:

$$\frac{\partial\phi_\alpha(x, t)}{\partial t} = -\frac{\delta S_E[\phi]}{\delta\phi_\alpha(x, t)} + \eta_\alpha(x, t) \tag{2}$$

where $\eta_\alpha(x, t)$ is a random noise with first moment zero and second moment

$$\langle\eta_\alpha(x, t)\eta_\beta(x', t')\rangle_\eta = 2\delta_{\alpha\beta}\delta^D(x - x')\delta(t - t') \tag{3}$$

The mean value $\langle F_{[\eta]}\rangle_\eta$ is defined over the Gaussian distribution of the η's.

The complete solution of equation (2) can be written in the following integral form:

$$\phi_\alpha(x, t) = \int d^D x' \int dt'\, G_{m^2}(x - x'; t - t')\left[\eta_\alpha(x', t') - \lambda\frac{\delta S_E^{int}[\phi]}{\delta\phi_\alpha(x', t')}\right] \tag{4}$$

where $G_{m^2}(x, t)$ is the free Green function:

$$G_{m^2}(x, t) = \theta(t)\int \frac{d^D p}{(2\pi)^D}\, e^{ipx-(p^2+m^2)t}$$

Following Greensite (1983) and Amundsen and Damgaard (1984), we use this last equation to build a variational approach to the solution of equation (2). A trial field is introduced which reads

$$\phi_\alpha^{[\sigma]}(x, t) = \int d^D x' \int dt'\, G_\sigma(x - x'; t - t')\eta_\alpha(x', t') \tag{5}$$

where $\sigma(p)$ is a functional parameter.

We then have

$$\lim_{t\to\infty} \mathrm{Tr}\langle(\phi_\alpha^{[\sigma]}(x, t))^2\rangle_\eta = N \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + \sigma(p)} \tag{6}$$

and clearly σ can be identified with the self-energy.

If we define

$$\tilde{\phi}_\alpha^{[\sigma]}(x, t) = \int d^Dx' \int dt'\, G_{m^2}(x - x'; t - t') \left[\eta_\alpha(x', t') - \lambda \frac{\delta S_E^{\text{int}}[\phi^{[\sigma]}]}{\delta\phi_\alpha^{[\sigma]}(x', t')} \right] \tag{7}$$

then minimizing

$$V_{[\sigma]} = \lim_{t\to\infty} \langle \mathrm{Tr}[\phi_\alpha^{[\sigma]}(x, t) - \tilde{\phi}_\alpha^{[\sigma]}(x, t)]^2\rangle_\eta \tag{8}$$

with respect to the variations of σ gives the best variational approximation to the solution of (2) in the subspace of our trial functions $\phi_\alpha^{[\sigma]}(x, t)$ given in (5).

It was shown in (Grandati et al. 1992, 1993) that the resulting self-energy can be built recursively using a $1/N$ expansion. This leads to the resolution of the following system:

$$\forall m \in \mathbb{N}, \qquad \sum_{n,k\geq 0} \sum_{\substack{i_1\cdots i_k\geq 1 \\ n+i_1+\cdots+i_k=m}} \frac{1}{k!} \int \left(\prod_{j=1}^{k} d^Dy_j\right) \sigma_{i_1}(y_1) \cdots \sigma_{i_k}(y_k)$$
$$\times \left. \frac{\delta^k F_n[\sigma]}{\delta\sigma(y_1) \cdots \delta\sigma(y_k)} \right|_{\sigma=\sigma_0} = 0 \tag{9}$$

where

$$F_n[\sigma(x)] = \frac{\delta V_n[\sigma]}{\delta\sigma(x)} \tag{10}$$

and V_n and σ_n are the nth-order terms in the expansions of V_σ and σ, respectively, in powers of $1/N$. Order by order, (9) is a Fredhom equation of the second kind:

$$\int d^Dy\, \sigma_n(y) \left. \frac{\delta F_0[\sigma(x)]}{\delta\sigma(y)} \right|_{\sigma=\sigma_0} = G(\sigma_0, \sigma_1(x), \ldots, \sigma_{n-1}(x)) \tag{11}$$

The solution of this equation allows for an iterative determination of all coefficients in the $1/N$ expansion of the mass operator $\sigma(p^2)$ once σ_0 is known. At the same time the determination of σ_m's of higher and higher order ensures a better and better trial field $\phi^{[\sigma]}$. The essential achievement of the method resides, for the $O(N)$ ϕ^4 theory, in arbitrary dimensions, in the analytical solution (Grandati *et al.*, 1993) of (11) for all order n in the expansion in $1/N$.

3. INTRODUCTION OF THE AUXILIARY FIELD: ϕ^4 THEORY

The method outlined above is now extended as follows. The Euclidean action reads

$$S_E[\phi] = \int d^D x \left[\frac{1}{2}\,\partial_\mu\phi_i(x)\partial_\mu\phi_i(x) + \frac{m^2}{2}\,\phi_i(x)\phi_i(x) + \frac{\lambda}{4!\,N}\,(\phi_i(x)\phi_i(x))^2\right] \tag{12}$$

where summation on repeated indices is implied. The common practice to eliminate the interaction term is to use the following identity in the partition function:

$$\int D\rho\,\exp\left[-\int d^D x\,\frac{N}{4!}\left(\rho(x) - \frac{ig}{N}\,\phi_i(x)\phi_i(x)\right)\right]^2 = \mathcal{N} \tag{13}$$

where $g = \sqrt{\lambda}$ and $\mathcal{N}$ is a number independent of $\phi_i(x)$.

In the partition function the action is now changed to

$$S_E[\phi, \rho] = \int d^D x \left[\frac{1}{2}\,\partial_\mu\phi_i\,\partial_\mu\phi_i + \frac{m^2}{2}\,\phi^2 + \frac{N}{4!}\,\rho^2 + ig\,\frac{\rho\phi^2}{12}\right] \tag{14}$$

where ϕ^2 stands for $\Sigma_i\,\phi_i(x)\phi_i(x)$.

In stochastic quantization, we have to consider both ϕ and ρ as coupled stochastic fields with generalized Langevin equations

$$\frac{\partial\phi_\alpha(x, t)}{\partial t} = -\int d^D y\,K_\phi(x - y)\,\frac{\delta S_E[\phi, \rho]}{\delta\phi_\alpha(y, t)} + \eta_\alpha(x, t) \tag{15}$$

$$\frac{\partial\rho(x, t)}{\partial t} = -\int d^D y\,K_\rho(x - y)\,\frac{\delta S_E[\phi, \rho]}{\delta\rho(y, t)} + \theta(x, t) \tag{16}$$

where η and θ are two independent Gaussian noise terms with two-point correlation functions given by

$$\langle\eta_\alpha(x, t)\eta_\beta(y, t')\rangle_\eta = 2K_\phi(x - y)\delta(t - t')\delta_{\alpha\beta} \tag{17}$$

$$\langle\theta(x, t)\theta^*(y, t')\rangle_\sigma = 2K_\rho(x - y)\delta(t - t') \tag{18}$$

In equations (15) and (16) two kernels K_ϕ and K_ρ are introduced to homogenize the dynamics for φ and ρ.

Convenient choices for K_ϕ and K_ρ are

$$K_\phi(x-y) = (\Box - m^2)^{-1}\delta^D(x-y), \qquad K_\rho(x-y) = \frac{12}{N}\delta^D(x-y)$$

In p-space the explicit forms of the Langevin equations become

$$\frac{\partial\phi_\alpha(p,t)}{\partial t} = -\phi_\alpha(p,t) - \frac{ig}{6}\frac{1}{p^2+m^2}\int\frac{d^Dp'}{(2\pi)^D}\rho(p',t)\phi_\alpha(p-p',t') + \eta_\alpha(p,t) \tag{19}$$

$$\frac{\partial\rho(p,t)}{\partial t} = -\rho(p,t) - \frac{ig}{N}\int\frac{d^Dp'}{(2\pi)^D}\phi_\alpha(p',t)\phi_\alpha(p-p',t) + \theta(p,t) \tag{20}$$

We define now the positive-definite quantity

$$V_\phi = \frac{1}{N}\lim_{t\to\infty}\mathrm{Tr}_\alpha\left\langle\left\|\int\frac{d^Dp}{(2\pi)^D}e^{ipx}[\phi_\alpha^{[\sigma]}(p,t) - \tilde{\phi}_\alpha^{[\sigma]}(p,t)]\right\|^2\right\rangle_{\theta,\eta} \tag{21}$$

and choose the trial field as

$$\phi_\alpha^{[\sigma]}(p,t) = \int_0^\infty dt'\, G_\sigma(p;t-t')\eta_\alpha(p,t') \tag{22}$$

with

$$G_a(p,t) = \theta(t)\exp\{-a(p)t\}$$

From equation (7) we have

$$\tilde{\phi}_\alpha^{[\sigma]}(p,t) = \int_0^t dt'\, e^{-(t-t')}\left[\eta_\alpha(p,t') - \frac{ig}{6}(p^2+m^2)^{-1}\right.$$
$$\left.\times\int\frac{d^Dp'}{(2\pi)^D}\rho^{[\beta,\gamma]}(p',t')\phi_\alpha^{[\sigma]}(p-p',t')\right] \tag{23}$$

To fix the form $\rho^{[\beta,\gamma]}$ of the trial composite field, we note that with the ansatz of equation (22) we have from equation (20)

$$\lim_{t\to\infty}\langle\rho(p,t)\rangle_{\theta,\eta} = i\beta(2\pi)^D\delta^D(p) \tag{24}$$

with

$$\beta = -g\int\frac{d^Dp}{(2\pi)^D}\frac{1}{\sigma(p)(p^2+m^2)} \tag{25}$$

From equation (20) one may also evaluate the "ρ-correlator" and find

$$\lim_{t\to\infty} \langle \rho(p, t)\rho^*(p', t)\rangle_{\theta,\eta}$$

$$= (2\pi)^{2D}\delta^D(p)\delta^D(p')\beta^2$$

$$+ (2\pi)^D\delta^D(p + p')\,\frac{12}{N}$$

$$\times \left[1 + \frac{g^2}{6}\int \frac{d^Dq}{(2\pi)^D}\,\frac{1}{\sigma(q)\sigma(p-q)[1 + \sigma(q) + \sigma(p-q)]}\right.$$

$$\left.\times\,\frac{1}{(q^2 + m^2)[(p-q)^2 + m^2]}\right] \tag{26}$$

Hence an ansatz for $\rho(p, t)$ modeled after the free-field solution of equation (20) and consistent with (24)–(26) is

$$\rho^{[\beta,\gamma]}(p, t) = i\beta(2\pi)^D\delta^D(p) + \int_0^\infty dt'\; G_\gamma(p; t - t')\theta(p, t') \tag{27}$$

With relation (18) one finds

$$\lim_{t\to\infty} \langle \rho^{[\beta,\gamma]}(p, t)\rho^{[\beta,\gamma]*}(p', t)\rangle_{\theta,\eta}$$

$$= (2\pi)^{2D}\delta^D(p)\delta^D(p')\beta^2 + (2\pi)^D\delta^D(p + p')\,\frac{12}{N}\,\frac{1}{\gamma(p)} \tag{28}$$

which fixes $\gamma(p)$ by comparison with equation (26).

The evaluation of the variational potential (21) is most easily performed using stochastic quantization diagrammatic rules (Bérard *et al.*, 1995; Grandati, 1991; Bérard, 1993) with the result

$$V_{[\phi]} = \int \frac{d^Dp}{(2\pi)^D}\,\frac{1}{p^2 + m^2}\left\{\frac{\{\sigma(p) - 1 + \beta g/[6(p^2 + m^2)]\}^2}{\sigma(p)[\sigma(p) + 1]}\right.$$

$$+ \frac{g^2}{3N(p^2 + m^2)}$$

$$\left.\times \int \frac{d^Dp'}{(2\pi)^D}\,\frac{1}{\gamma(p')\sigma(p - p')[\gamma(p') + \sigma(p - p') + 1][(p - p')^2 + m^2]}\right\} \tag{29}$$

In the large-N limit, and defining

$$\sigma(p) = \frac{p^2 + \Sigma(p)}{p^2 + m^2} \tag{30}$$

V_ϕ is minimum if Σ_0 is a solution of the usual gap equation

$$\Sigma_0 = m^2 + \frac{g^2}{6} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + \Sigma_0} \tag{31}$$

For N finite, minimizing V_ϕ with respect to the variation of $\sigma(p)$ gives an integral equation which is solved by a $1/N$ expansion according to the procedure of Section 2. As shown for the toy model of the next section, the variational procedure for $\sigma(p)$ is very efficient even for the crudest approximation $\gamma(p) = 1$. In this case the expression given in Grandati *et al.* (1992) for $\sigma_1(p^2)$ becomes quite simple,

$$\sigma_1(p) = \frac{g^2\sigma_0(1 + \sigma_0)}{6(1 + 2\sigma_0)^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[(p - q)^2 + m^2]} \times \left[\frac{1}{[(p - q)^2 + m^2]} + \frac{p^2 + m^2}{(q^2 + m^2)^2}\right]$$

In the procedure above one may wonder why the ϕ-field was not eliminated completely from the outset to leave a ρ-dependent effective action only, and then a single Langevin equation for this field. Although this would lead to the correct ρ-propagator to leading order in $1/N$, the study of symmetry breaking is only possible with at least one component of the ϕ-field kept. In this perspective we choose here to keep all of them. The benefit is a simple Langevin equation for ρ [equation (20)] which can be readily integrated. However, the price to pay is a ρ-propagator which, with the ansatz of equation (22), matches the exact one only in the perturbative regime. We show in the sequel that this is not detrimental to a good determination of the physical ρ-propagator.

4. APPLICATION: TOY MODEL

In the case $D = 0$, (30) becomes

$$V_\phi = \frac{[\sigma - 1 - g^2/(6m^4\sigma)]^2}{m^2\sigma(\sigma + 1)} + \frac{1}{N} \frac{g^2}{3m^6\gamma\sigma(\sigma + \gamma + 1)} \tag{32}$$

with $\gamma = 1 + g^2/[6m^4\sigma^2(1 + 2\sigma)]$.

In the large-N limit, $\sigma_0 = 1 + g^2/(6m^4\sigma_0)$ and with $\Sigma_0 = m^2\sigma_0$ one finds (Grandati *et al.*, 1993)

$$\frac{1}{N}\langle\phi^2\rangle = \frac{1}{\Sigma_0} = \frac{1}{(m^2/2)(1 + \sqrt{1 + 2g^2/3m^4})} \tag{33}$$

The direct minimization of (32) is straightforward. In Fig. 1 we plot different estimations of $\langle\phi^2\rangle/N$ [exact, saddle point, and global variational results with and without (Grandati *et al.*, 1993) the auxiliary field] as a function of log(g) for $N = 3$. Figure 2 shows the difference $\Delta\langle\phi^2\rangle = (1/N)(\langle\phi^2\rangle_\gamma - \langle\phi^2\rangle_{\gamma=1})$ as a function of log(g) for $N = 3$. It is clear that even for the crudest approximation of the ρ-propagator corresponding to $\gamma = 1$ the correlation $\langle\phi^2\rangle/N$ is noticeably improved over the whole range of coupling with respect to the procedure with no auxiliary field. For $\gamma = 1$ the variational first-order correction in $1/N$ is particularly simple and reads, with $u = (3m^4/g^2)^{1/2}$,

$$\sigma_1 = \frac{\sigma_0(3u + \sqrt{2 + u^2})^2}{(u^2 + 2)(5u + \sqrt{2 + u^2})^2} \tag{34}$$

The correlator is then

$$\frac{\langle\phi^2\rangle}{N} = \frac{1}{m}\frac{2u}{u + \sqrt{2 + u^2}}\left(1 - \frac{1}{N}\frac{a_2(u)}{(2 + u^2)}\right) + O\left(\frac{1}{N^2}\right) \tag{35}$$

with

$$a_2(u) = \left(\frac{3u + \sqrt{2 + u^2}}{5u + \sqrt{2 + u^2}}\right)^2 \tag{36}$$

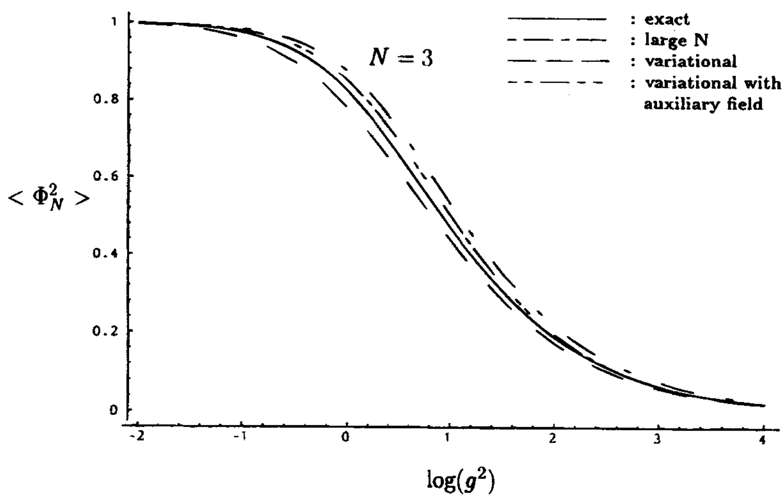

Fig. 1. Propagator $\langle\phi^2\rangle/N$ as a function of the coupling g for $N = 3$. (—) Exact solution (Grandati *et al.*, 1993); (- -) global variational approximation of the self-energy without the auxiliary field (Grandati *et al.*, 1993); (-----) global variational approximation of the self-energy with the auxiliary field; (---) large N-limit.

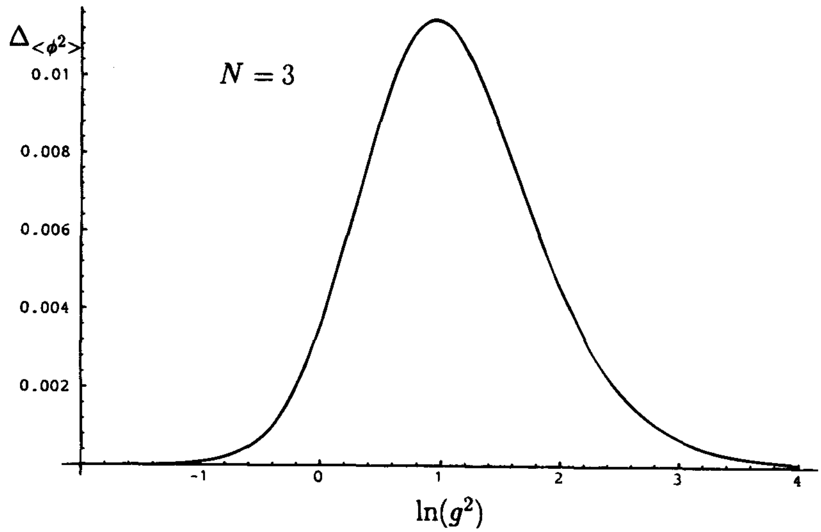

Fig. 2. The difference $\Delta\langle\phi^2\rangle$ of the propagators $\langle\phi^2\rangle/N$ taken with $\gamma = 1 + g^2/[6m^4\sigma^2(1 + 2\sigma)]$ and $\gamma = 1$ as a function of the coupling g for $N = 3$.

Even though the precise value of the ρ-propagator Δ_ρ does not seem essential in determining the variational estimate of the two-point correlation function from the minimization of equation (32), it is of interest to test its approximations since, being related to the ϕ-field four-point function, it governs the coupling constant renormalization. To leading order in $1/N$ the ρ-propagator $\Delta_{\rho,N}$ is known (Zinn-Justin, 1990) to be

$$\Delta_{\rho,N}(p) = \frac{12}{N}\left[1 + \frac{g^2}{6}\int \frac{d^Dq}{(2\pi)^D} \frac{2}{(q^2 + \Sigma_0)((p - q)^2 + \Sigma_0)}\right]^{-1} \tag{37}$$

In Fig. 3 we compare, for $D = 0$, different approximations to Δ_ρ [exact, $\Delta_{\rho,N}$, variational deduced from equation (20)] as a function of the coupling g. It is clear that keeping all components of the ϕ-field with a general linear ansatz in the noise η according to equation (22) fails to reproduce the ρ-propagator beyond the perturbative regime. This is not surprising, as a linear ansatz in the Gaussian noise η cannot be expected to build up correctly a correlation function beyond the two-point level.

5. CONCLUSION

In its original form the stochastic variational principle of Greensite (1983) and Amundsen and Damgaard (1984) aims at a nonperturbative deter-

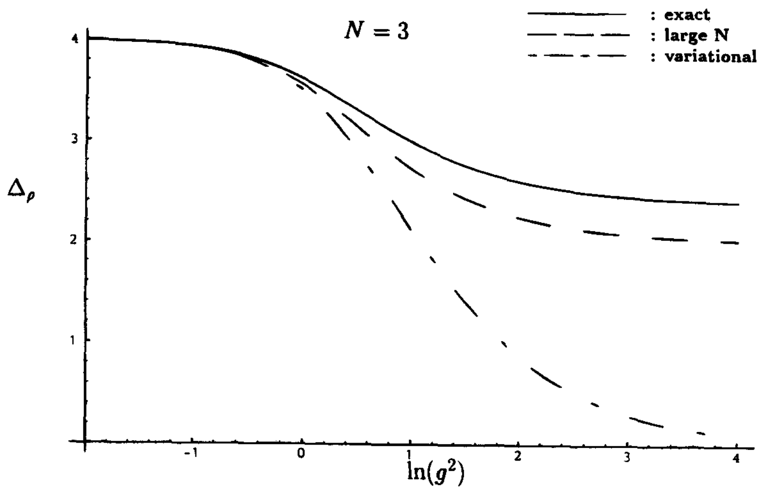

Fig. 3. The propagator Δ_ρ of the auxiliary field as a function of the coupling g for N = 3. (—) Exact solution (Bérard *et al.*, 1996); (– –) solution to leading order in $1/N$; (---) variational estimate.

mination of the self-energy. In practice it can only be implemented through a $1/N$ expansion, as shown in Grandati *et al.* (1992, 1993). Although arguments can be given to assess the good convergence properties of the self-energy expansion in $1/N$ in this framework, the question remained, on the one hand, of improving the variational scheme and, on the other hand, of access to nontrivial correlation functions of higher order than 2. We have shown here that these goals can be partially achieved with the introduction of an auxiliary composite field with its own stochastic dynamic. The improved effectiveness of the variational principle including this composite object can be attributed to the incorporation of nonlinear effects through its parametrized propagator. They could not be taken into account in the formalism involving trial ϕ-fields alone, linear in the noise. However, the composite field propagator obtained here is only valid in the perturbative regime in the coupling constant and hence is improper for renormalization studies. To cure this deficiency one has to operate from the outset with an effective action where all but one component of the ϕ-field have been integrated out, as we shall report elsewhere (Bérard *et al.*, 1996).

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